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On the Accuracy of Parabolic Wave Models

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Abstract

Errors in parabolic equation models (PEM) for wave refraction and diffraction are investigated by examining the case of waves propagating over a planar bathymetry for which the analytical solution is well known. The models investigated are: i) lowest order parabolic approximation (Simple method), ii) Padé approximation and iii) Minimax approximation models.

Errors in wave heights, wave directions, radiation stresses and resulting longshore currents are investigated analytically and by numerical tests using the parabolic equation model MIKE 21 PMS. The results indicate that, while the predicted wave directions are generally accurate, the wave heights, radiation stresses and longshore currents can contain significant errors depending on the parabolic approximation used.

Introduction

Parabolic equation models (PEM) for wave refraction/diffraction are frequently used in coastal engineering practice for the computation of wave parameters, radiation stresses and associated wave-induced nearshore currents in coastal areas. The PEM approach is attractive since it is computationally more efficient than the complete elliptic mild-slope model, and it provides useful engineering results for refraction/diffraction problems (in the absence of significant reflection) in the coastal area (Berkhoff et. al., 1982, Johnson et. al., 1994).

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The parabolic equation model is derived as an approximation to the elliptic mild slope equation (derived by Berkhoff, 1972) governing the refraction, shoaling, diffraction and reflection of linear water waves propagating on mildly sloping bathymetry. The lowest order parabolic approximation is obtained by assuming a principal wave direction (x-direction) and neglecting backscatter. Kirby (1986) extended this approximation to allow waves propagating at large angles to the assumed principal direction. Many numerical models have been developed based on these approximations. One such wave model is the MIKE 21 Parabolic Mild Slope (PMS) model, which is based on the equations derived by Kirby (1986).

As a result of the underlying assumptions, PEM do not exactly reproduce the refraction coefficient for waves approaching the coast at an angle. This introduces errors in the nearshore wave heights and radiation stresses and, consequently, in the wave-driven longshore currents. The magnitude of the errors depends on the type of parabolic approximation.

In this paper, we attempt to quantify the errors and give suggestions for minimizing the influence of the errors in real applications. This is achieved by examining the simple case of wave propagation over straight and parallel depth contours for which an analytical solution exists.

This paper is broadly divided into two parts. In the first part, the PEM is used to derive evolution equations for wave heights and directions over straight and parallel contours. This is compared with the analytical solution, and thus the errors in the PEM are obtained. In the second part, results of numerical tests using MIKE 21 PMS are compared with the analytical solution in order to quantify the errors in the various parabolic approximations. Finally, some conclusions are derived regarding an optimal PEM for practical situations.

Parabolic Equation Model

The starting point for the analysis is the general parabolic equation derived by Kirby (1986) for linear waves in the absence of currents. Kirby removed the rapidly varying term from the mean free surface potential ϕ, to obtain a slowly varying complex function A (see Eq. 1), which is used as the primary variable in the parabolic equation

$$\phi = A(x, y)e^{ik_0 x} \tag{1}$$

In Eq. 1, k_0 is a characteristic wave number, typically chosen as the average wave number along the y-direction, and x is the principal wave direction. Kirby obtained the parabolic equation given in Eq. 2.

$$A_x + i(k_o - \beta_1 k)A + \frac{A}{2c_g}(c_g)_x + \frac{\sigma_1}{\omega c_g}(cc_g A_y)_y$$
$$+ \frac{\sigma_2}{\omega c_g}(cc_g A_y)_{yx} + \frac{W}{2c_g}A = 0 \qquad (2)$$

where

$$\sigma_1 = i\left(\beta_2 - \beta_3 \frac{k_o}{k}\right) + \beta_3\left(\frac{k_x}{k^2} + \frac{(c_g)_x}{2kc_g}\right) \qquad (3)$$

$$\sigma_2 = -\frac{\beta_3}{k} \qquad (4)$$

The subscripts x, y respectively represent differentiation with respect to x and y, i is the imaginary unit, k is the local wave number, c_g is the local group velocity, ω is the circular wave frequency, β_1, β_2 and β_3 are the coefficients of the parabolic approximation, and W is a dissipation term (e.g. bottom dissipation, wave breaking). The β-coefficients are given in Table 1 for various parabolic approximations.

Approximation	β_1	β_2	β_3
Simple	1	-0.5	0
Padé (1,1)	1	-0.75	-0.25
Minimax 10°	0.999999972	-0.752858477	-0.252874920
Minimax 20°	0.999998178	-0.761464683	-0.261734267
Minimax 30°	0.999978391	-0.775898646	-0.277321130
Minimax 40°	0.999871128	-0.796244743	-0.301017258
Minimax 50°	0.999465861	-0.822482968	-0.335107575
Minimax 60°	0.998213736	-0.854229482	-0.383283081
Minimax 70°	0.994733030	-0.890064831	-0.451640568
Minimax 80°	0.985273164	-0.925464479	-0.550974375

Table 1. Coefficients for various parabolic approximations.

The lowest order parabolic approximation is denoted as the Simple approximation in Table 1. The Minimax approximations minimize the error in the wave number over the entire aperture width associated with it. However, this does not preclude that the error for specific directions (within the aperture width) may be large.

Parabolic Equation Model for Straight and Parallel Depth Contours

The evolution equations for wave heights and directions are derived here for the case of straight and parallel depth contours and no dissipation (W = 0). The Cartesian co-ordinate system is chosen such that the y-axis is parallel to the coastline.

The mean free surface potential for plane, progressive waves can be written as:

$$\phi = A^*(x,y)e^{i\Psi} \quad ; \qquad \Psi = \int_x k\cos\theta\, dx + \int_y k\sin\theta\, dy \tag{5}$$

where $A^*(x,y)$ is the amplitude function (equals half of the wave height H) and Ψ is the phase function. Combining Eqs. 1 and 5, the slowly varying mean surface potential can be expressed as:

$$A = A^*(x,y)e^{i(\Psi - k_0 x)} = \frac{H}{2}(x,y)e^{i(\Psi - k_0 x)} \tag{6}$$

Thus, the derivatives in Eq. 2 can be found as a function of $A^*(x,y)$ and Ψ. For example, A_y is obtained as:

$$A_y = \left\{A^*(x,y)\cdot i\Psi_y + A^*_y\right\}e^{i(\Psi - k_0 x)} \tag{7}$$

For straight and parallel depth contours, the wave height H, wave number k, phase speed c, and group velocity c_g, are all constant along the contours (y-direction). Introducing these constraints and rearranging, Eq. 2 gives:

$$A_x = \frac{\left\{i(\beta_1 k - k_0) - \left[\frac{(c_g)_x}{2c_g} - k\sigma_1\sin^2\theta - \frac{k^2\sin^2\theta}{\omega c_g}\sigma_2(cc_g)_x\right]\right\}}{1 - k\sigma_2\sin^2\theta}A \tag{8}$$

Using Eq. 6, an expression for A_x/A in terms of the wave height and phase can be written as:

$$\frac{A_x}{A} = \frac{H_x}{H} + i(\Psi_x - k_0) = \frac{H_x}{H} + i(k\cos\theta - k_0) \tag{9}$$

Equating the real and imaginary parts of Eq. 9 with those of Eq. 8, the following evolution equations for wave height H and direction θ are derived from the PEM

$$\frac{H_x}{H} + \frac{(c_g)_x}{2c_g} - 2\beta_3\frac{S^2/k^2}{(1+\beta_3 S^2/k^2)}\frac{k_x}{k} = 0 \tag{10}$$

$$\cos\theta = \left(\frac{\beta_1 + \beta_2 S^2 / k^2}{1 + \beta_3 S^2 / k^2} \right) \tag{11}$$

where $S = k \sin\theta$.

Exact solution for straight and parallel depth contours

The evolution equations for the exact solution are derived from Snell's law and the conservation of energy flux. According to Snell's law, $k \sin\theta = \text{constant} = S$. This can be alternatively written as:

$$\cos\theta = \sqrt{1 - \left(\frac{S}{k} \right)^2} \tag{12}$$

Eq. 12 may be accurately approximated as (see Figure 1):

$$\cos\theta = \left(\frac{\beta_1 + \beta_2 S^2 / k^2}{1 + \beta_3 S^2 / k^2} \right) \tag{13}$$

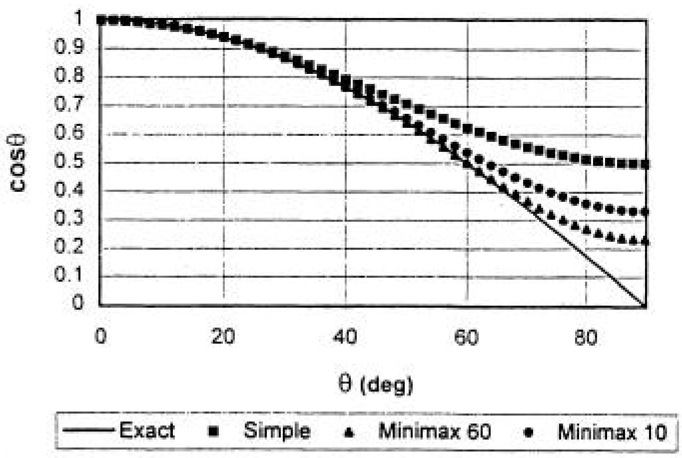

Figure 1. Approximate expressions for $\cos\theta$ using β coefficients for Simple, Minimax 10° and Minimax 60° PEM.

The conservation of energy flux equation is written as:

$$H^2 c_g \cos\theta = \text{constant} \tag{14}$$

Differentiating Eq. 14 with respect to x, the exact evolution equation for the wave height is obtained as:

$$\frac{H_x}{H} + \frac{(c_g)_x}{2c_g} + \frac{(\cos\theta)_x}{2\cos\theta} = 0 \tag{15}$$

Inserting Eq. 13 (using Minimax 60° coefficients, indicated by *) into Eq. 15, an almost exact evolution equation (Eq. 16) for wave height is obtained.

$$\frac{H_x}{H} + \frac{(c_g)_x}{2c_g} + \frac{\beta_1^*\beta_3^* - \beta_2^*}{(\beta_1^* + \beta_2^* S^2/k^2)} \frac{S^2/k^2}{(1+\beta_3^* S^2/k^2)} \frac{k_x}{k} = 0 \tag{16}$$

Eq. 16 can be readily compared with the corresponding evolution equation from the PEM, Eq. 9.

Errors

Since Eqs. 11 and 13 are identical, it is clear that the errors in wave directions are small in PEM, as long as Eq. 13 is a good approximation to Eq. 12. This covers a wide range of directions (as shown in Fig. 1), even for the lowest order approximation. Thus, calculated wave directions from PEM would be generally accurate.

Comparing Eqs. 10 and 16, it is obvious that the PEM evolution equation for the wave height is different from the analytical solution, resulting in errors.

In order to relate the errors to physically meaningful properties such as the shoaling coefficient K_s, and refraction coefficient K_r, the wave height at an inshore location is written as:

$$H = H_0 K_s K_r \tag{17}$$

where H_0 is the wave height at an offshore location. Differentiating Eq. 17 with respect to x and dividing through by $H = H_0 K_s K_r$ gives:

$$\frac{H_x}{H} - \frac{K_{s,x}}{K_s} - \frac{K_{r,x}}{K_r} = 0 \tag{18}$$

Comparing Eqs. 18 and 15, it is clear that the second term in Eq. 15 (and in Eqs. 10 and 16) is related to the shoaling coefficient, while the third term is related to the refraction coefficient. Thus, while the shoaling coefficient is exactly reproduced in PEM, the refraction coefficient in PEM contains some errors. The refraction terms in Eqs. 10 and 16 are reproduced below:

$$\left[\frac{K_{r,x}}{K_r}\right]_{PEM} = \frac{2\beta_3 S^2/k^2}{1+\beta_3 S^2/k^2} \frac{k_x}{k} \tag{19}$$

$$\left[\frac{K_{r,x}}{K_r}\right]_{exact} = \frac{\left(\beta_2^* - \beta_1^*\beta_3^*\right)}{\left(\beta_1^* + \beta_2^* S^2/k\right)} \frac{S^2/k^2}{\left(1+\beta_3^* S^2/k^2\right)} \frac{k_x}{k} \qquad (20)$$

For waves approaching perpendicular to the coast (θ = 0), S = 0, and both refraction terms are zero. Thus, in this special case, the wave heights in the PEM are calculated correctly, since there is no refraction. The same is true for constant water depth where $k_x = 0$.

It is also seen that the refraction term in the parabolic equation model is identically zero for all angles if the Simple approximation is used, since $\beta_3 = 0$. Thus, the Simple approximation does not account for refraction effects at all.

For oblique incidence (and varying depth), we define λ as the ratio of the refraction term in the PEM to that in the exact solution.

$$\lambda = 2\beta_3 \frac{\left(\beta_1^* + \beta_2^* S^2/k^2\right)}{\beta_2^* - \beta_1^*\beta_3^*} \frac{\left(1+\beta_3^* S^2/k^2\right)}{\left(1+\beta_3 S^2/k^2\right)} \qquad (21)$$

The ideal situation is $\lambda = 1$. This corresponds to the case where the refraction term is exactly reproduced in PEM. For $\lambda < 1$, the refraction term is underestimated in PEM (i.e. not enough refraction), while for $\lambda > 1$, the refraction term is over-exaggerated (i.e. too much refraction). In the case of the Simple approximation, the refraction term is not reproduced at all, i.e. λ=0. In Figure 2, $(\lambda - 1)$ is shown as a function of wave direction θ for various parabolic approximations given in Table 1 (except Simple).

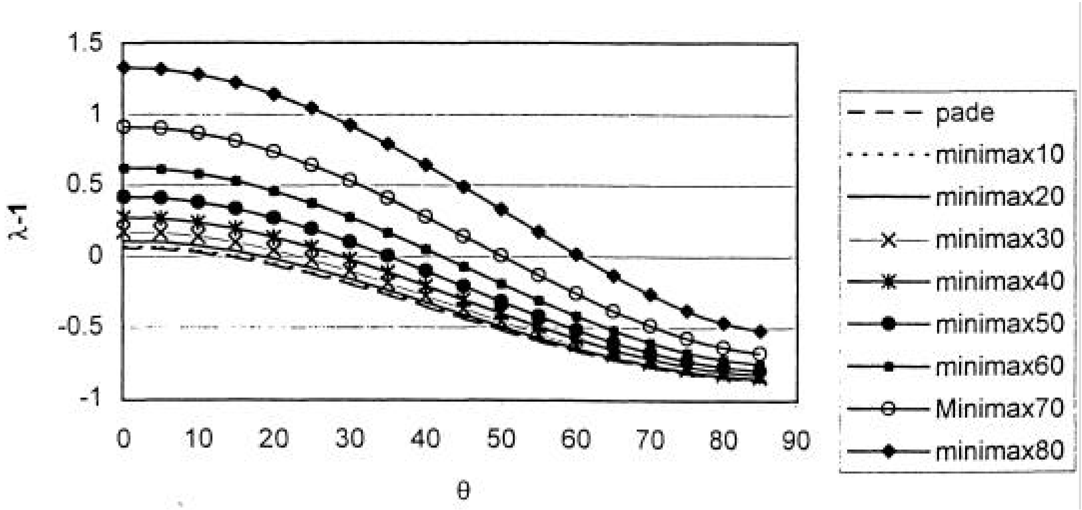

Figure 2. Error in the refraction term as a function of incident wave direction for various parabolic approximations.

It is seen from Figure 2 that the Padé approximation gives the minimum error for small angles of incidence, 0° to 20°. For larger angles, the Minimax 50° or Minimax 60° give the minimum error on average.

Using Eq. 18, we write:

$$\left(\frac{H_x}{H}\right)_{PEM} = \left(\frac{(K_s)_x}{K_r}\right)_{exact} + \lambda\left(\frac{(K_r)_x}{K_r}\right)_{exact} \tag{22}$$

$$\left(\frac{H_x}{H}\right)_{exact} = \left(\frac{(K_s)_x}{K_r}\right)_{exact} + \left(\frac{(K_r)_x}{K_r}\right)_{exact} \tag{23}$$

Using Eqs. 22 and 23, the error in wave height obtained with the PEM at a distance Δx from the offshore location is obtained as:

$$H_{\Delta x,PE} - H_{\Delta x,exact} = (\lambda - 1)H_0 \Delta x \cdot \left(\frac{(K_r)_x}{K_r}\right)_{exact} \tag{24}$$

The refraction coefficient $K_r = \sqrt{\cos\theta_i / \cos\theta_0}$, where θ_i is the inshore wave direction which decreases towards the shore (with increasing x), hence $(K_r)_x/K_r$ is always less than 0. Thus, it follows that if $\lambda > 1$, the inshore wave height will be underestimated (since the refraction effect is over-exaggerated). The converse is true for λ < 1. For λ = 1, the error is identically zero. However, there is no PEM that gives λ = 1 for any range of wave directions. For example, if Padé approximation is used, the parabolic equation model will overestimate the inshore wave heights for most wave directions (> 15°).

<u>Radiation stresses</u>

The next important question is what effect these errors have on wave-driven longshore currents calculated using the parabolic equation model. For straight and parallel depth contours, the important radiation stress gradient term for these currents is

$$\left(\frac{\partial S_{xy}}{\partial x}\right)_{exact} = \frac{\sin\theta}{c}\frac{\partial E c_g \cos\theta}{\partial x} = S_{xy}\left\{2\frac{H_x}{H} + \frac{(c_g\cos\theta)_x}{c_g\cos\theta}\right\}_{exact} \tag{25}$$

Outside the surf zone (disregarding bottom friction), the right-hand side should be zero, since there is no dissipation. Performing the differentiation of the second term in the curly braces and rearranging, we obtain:

$$\frac{1}{S_{xy}}\left(\frac{\partial S_{xy}}{\partial x}\right)_{exact} = 2\left(\frac{H_x}{H}\right)_{exact} + \left(\frac{(\cos\theta)_x}{\cos\theta}\right)_{exact} + \left(\frac{(c_g)_x}{c_g}\right)_{exact} \tag{26}$$

A similar expression is obtained for the parabolic equation model. The last term in Eq. 25 is the shoaling term, which is identical both in the exact and the parabolic case. The second term is the refraction term.

Using Eqs. 22 and 23, and remembering that $(\cos\theta)_x/\cos\theta = (K_r)_x/K_r$, the difference between the exact and the PEM expression for the radiation stress term is obtained as shown below:

$$\begin{aligned} &\frac{1}{S_{xy}}\left(\frac{\partial S_{xy}}{\partial x}\right)_{PEM} - \frac{1}{S_{xy}}\left(\frac{\partial S_{xy}}{\partial x}\right)_{exact} \\ &= 2\left\{\left(\frac{H_x}{H}\right)_{PEM} - \left(\frac{H_x}{H}\right)_{exact}\right\} + \left(\frac{(\cos\theta)_x}{\cos\theta}\right)_{PEM} - \left(\frac{(\cos\theta)_x}{\cos\theta}\right)_{exact} \\ &= 2(\lambda-1)\left(\frac{K_{r,x}}{K_r}\right)_{exact} + 2(\lambda-1)\left(\frac{K_{r,x}}{K_r}\right)_{exact} = 4(\lambda-1)\left(\frac{K_{r,x}}{K_r}\right)_{exact} \end{aligned} \tag{27}$$

Outside the surf zone, we therefore see that:

$$\frac{1}{S_{xy}}\left(\frac{\partial S_{xy}}{\partial x}\right)_{PE} = 4(\lambda-1)\left(\frac{K_{r,x}}{K_r}\right)_{exact} \tag{28}$$

which is generally non-zero, meaning that currents are generated which should not be there. Only when $S = 0$ (or the depth is constant) is the right-hand side zero, as it should be, for all wave directions. The type of current generated depends on λ. If $\lambda > 1$, $\partial S_{xy}/\partial x < 0$, and spurious currents are generated which may look realistic. On the other hand, when $\lambda < 1$, $\partial S_{xy}/\partial x > 0$ and clearly unrealistic currents are generated in the "opposite" direction to wave propagation.

In the following second part, numerical examples are presented in order to quantify the errors.

Numerical Examples

The parabolic approximations implemented in MIKE 21 PMS will now be compared with the analytical solution on a beach with straight and parallel depth contours. The slope is 1:50 and the water depth at the toe of the slope is 10m. The shore is parallel to the y-axis and the wave direction is measured from the x-axis. A monochromatic wave is considered, with wave height H=1m at 10m depth and wave period T=8s.

The results in Figures 3 to 5 are shown as functions of the wave direction at the toe of the slope, for a location halfway up the slope, where the depth is 5m. Note that the errors generally increase with decreasing water depth. However, in very shallow water, when wave breaking becomes important, the energy dissipation due to wave breaking is likely to be more important than the refraction error in the PEM. This aspect was not investigated in this study.

In Figure 3, the relative errors (in %) in the wave directions calculated using the parabolic equation model are shown. It is seen that the errors are less than 1%, even for directions as large as 70°. This confirms that wave directions are calculated accurately using the parabolic equation model.

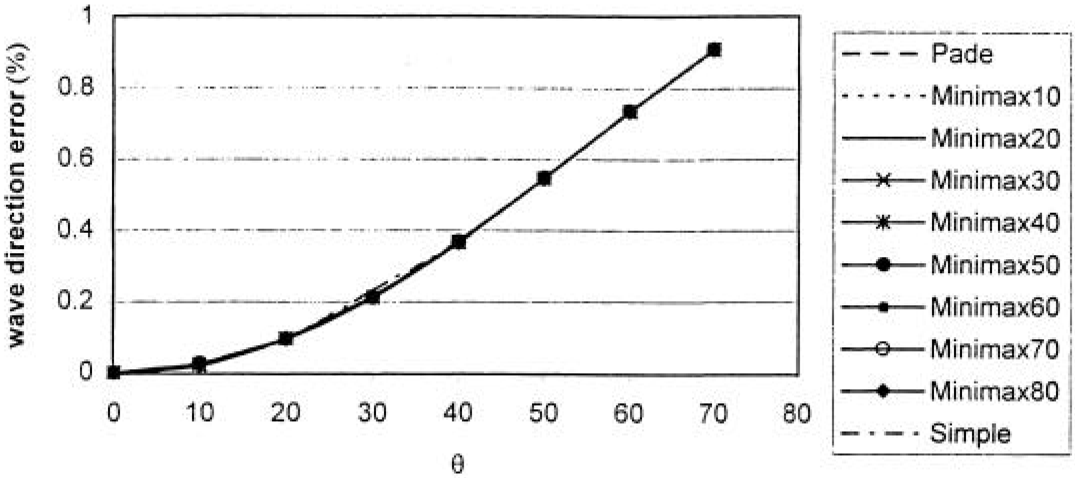

Figure 3. Relative errors in wave directions at 5m depth.

In Figure 4, the corresponding relative errors in wave heights are shown. As expected, these errors are much larger, especially for the Simple approximation. For very large wave directions (directions > 50°), the Minimax 80° performs well, however, it produces much larger errors for smaller directions. Generally, the various approximations are very accurate (less than 5% error) for directions up to 50°.

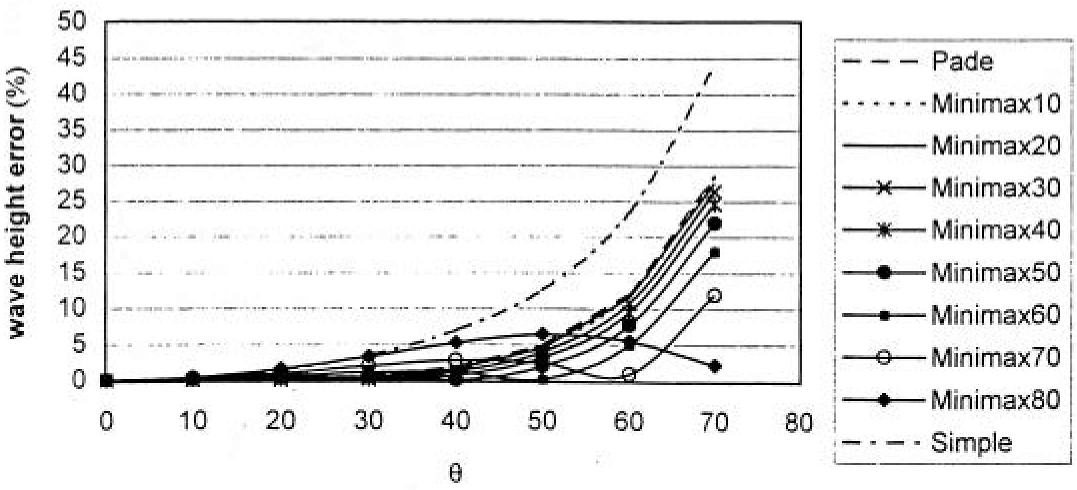

Figure 4. Relative errors in wave heights at 5m depth.

Figure 5 shows the relative errors in the radiation stress term S_{xy}. Notice that it is not the radiation stress *gradient* that is shown. It is again seen, that the Simple approximation only works well for very small directions and that the Padé approximation and the Minimax 10°-30° approximations are almost identical. The errors for larger wave directions decrease with increasing aperture width in the Minimax approximation. However, the Minimax approximation with large aperture width (Minimax 70°, Minimax 80°) gives bigger errors than the other PEM for small wave directions.

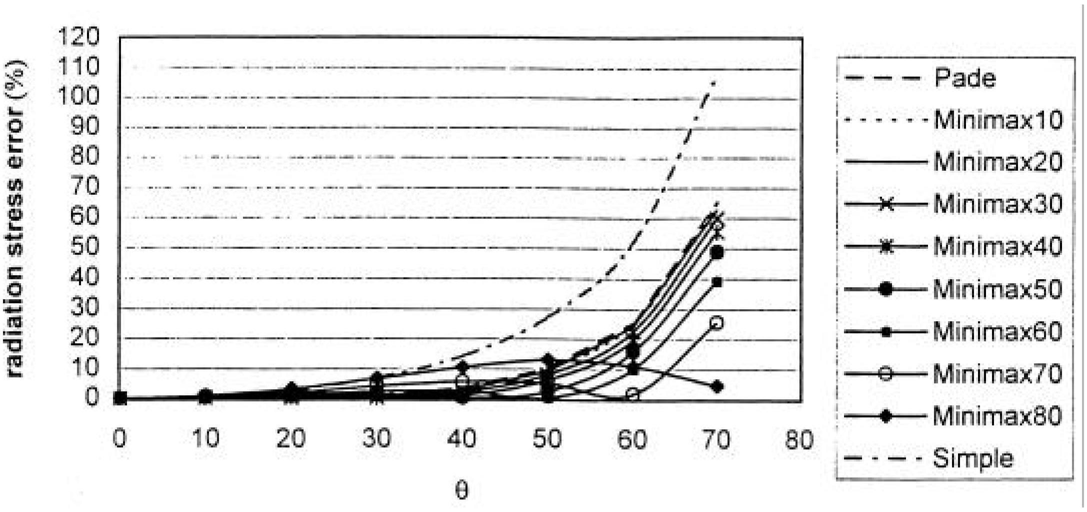

Figure 5. Relative errors in the radiation stress term S_{xy} at 5m depth.

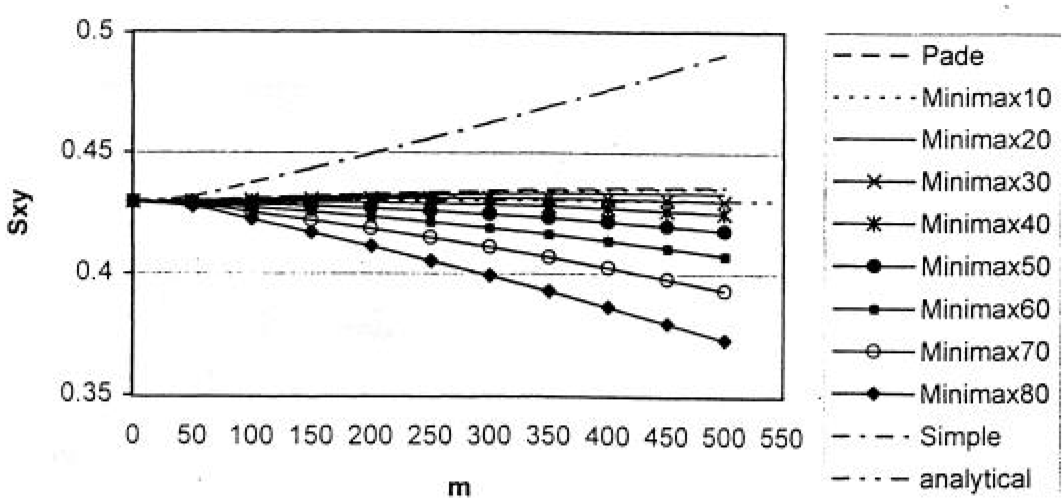

Figure 6. Radiation stresses S_{xy} as a function of distance towards the shore (x). The wave direction at the toe of the slope is 30°.

In Figure 6, S_{xy} is shown as a function of the distance towards the shore (x) for the case where the wave direction at the toe of the slope is 30°. From this figure, we can see the magnitude of the gradient $\partial S_{xy}/\partial x$ that generates the false currents in the parabolic equation model. The Simple approximation has a large positive gradient, which gives clearly unrealistic currents. The Padé and the Minimax 10-30 approximations have very small gradients. These gradients will not generate significant currents. The higher Minimax approximations generate spurious currents in this case because the gradients are negative.

The magnitude of the currents can be seen in Figure 7 where the longshore current profiles are given for four cases. The first three show the performance of the Simple approximation, the Minimax 40° and the Minimax 80° approximation for an angle of incidence of 30^o. The Minimax 40° approximation is very good in this case, there is almost zero current outside the surf zone, whereas the two others generate modest currents. The fourth case shows the current profile for the Simple approximation for the case of a large incident wave direction (60°). In this case, the false current offshore of the surf zone is seen to be of a significant magnitude compared with the surf zone current.

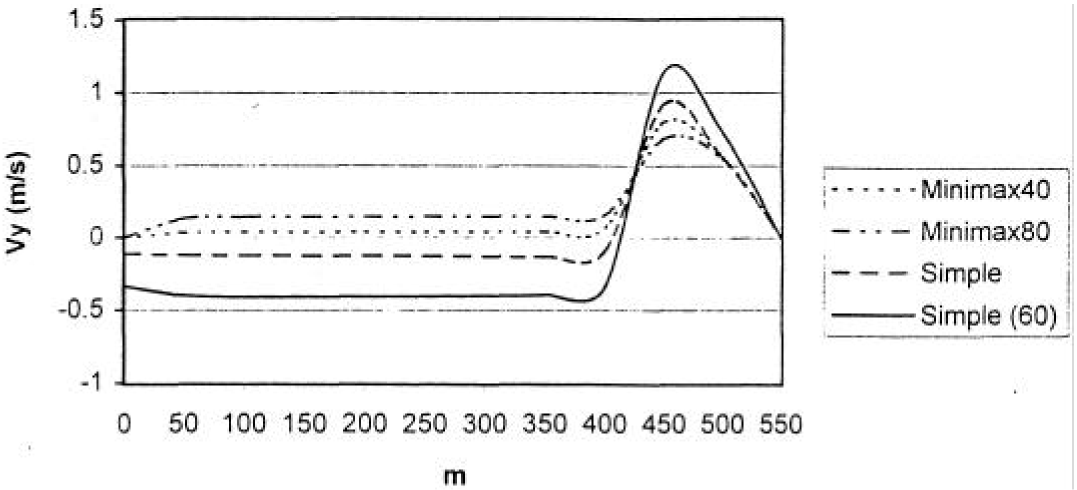

Figure 7. Longshore current profiles.

Conclusions

The investigations in this paper show that the predicted wave directions from PEM models are generally accurate. However, wave heights, radiation stresses and longshore currents predicted from PEM can contain significant errors depending on the parabolic approximation used. The error arises because of the inexact representation of the refraction coefficient in PEM.

In the examples given here, it was easy to quantify the errors and identify the false longshore currents. However, for a complex bathymetry as usually encountered in nature, it will generally not be as easy, and we must emphasize the importance of making a proper choice of parabolic approximation. Also, in real applications the incident wave field will be directional, and thus, what works for one wave direction may not work for another.

Definitely, the Simple approximation should not be used in cases where the waves approach at an angle. In most practical cases, using a Minimax 50° or 60° approximation should work well, since this gives, on the average, a small error in the refraction coefficient over a large range of wave directions (0° to 60°) that will normally be encountered in practice. For cases where the wave directions in the study area are known a-priori to be less than 30°, the Padé approximation is recommended.

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